Cross sections and decay rates

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Normalization of states

non-relativistic : box of volume $V = L^3$, periodic boundary conditions

$$\Downarrow$$

$$\bar{p} = \frac{2\pi}{L}\,\bar{n}\,, \qquad \bar{n} = (n_p^x, n_p^y, n_p^z)$$

1-particle state of momentum $\bar{p}$: $\psi_p(x) = C\,e^{i\bar{p}\cdot\bar{x}}$

1-particle in the volume : $\int_V |\psi_p(x)|^2 \, d^3x = 1$

$$\Downarrow$$

$$\psi_p(x) = \frac{1}{V^{1/2}}\, e^{i\bar{p}\cdot\bar{x}}$$

$$\int_V \psi_p^*(x)\,\psi_q(x)\,d^3x = \delta_{\bar{p},\bar{q}} \equiv \delta_{n_p^x, n_q^x}\,\delta_{n_p^y, n_q^y}\,\delta_{n_p^z, n_q^z}$$

$$\psi_p(x) = \langle x | p \rangle \quad \& \quad \int d^3x\, |x\rangle\langle x| = 1$$

$$\Downarrow$$

$$\langle p_1 | p_2 \rangle^{(NR)} = \int d^3x\, \langle p_1 | x \rangle \langle x | p_2 \rangle = \int d^3x\, \psi_{p_1}^*(x)\,\psi_{p_2}(x) = \delta_{\bar{p}_1, \bar{p}_2}$$

non-relativistic

relativistic

— non-relativistic

relativistic

normalization $\int_V |\psi_p(x)|^2 d^3x = 1$ is not convenient since V is not Lorentz invariant

instead we adopt $\langle p_1 | p_2 \rangle^{(R)} = 2 E_{p_1} V \delta_{\bar{p}_1, \bar{p}_2}$

$$\Downarrow$$

$$|p\rangle^{(R)} = (2 E_p V)^{1/2} \, |p\rangle^{(NR)}$$

for multi-particle states

$$|p_1, \dots, p_n\rangle^{(R)} = \left(\prod_{i=1}^{n} (2 E_{p_i} V)^{1/2} \right) |p_1, \dots, p_n\rangle^{(NR)}$$

$$\underset{(NR)}{M_{fi}} = \underbrace{\left(\prod_{i=1}^{n} (2 E_{p_i} V)^{-1/2} \right)}_{\text{final state}} \underbrace{\left(\prod_{j=1}^{m} (2 E_{p_j} V)^{-1/2} \right)}_{\text{initial state}} \underset{(R)}{\mathcal{M}_{fi}}$$

$$S = \mathbb{1} + iT$$

$$S_{fi} = \delta_{fi} + \underbrace{(2\pi)^4 \delta^{(4)}(P_i - P_f) \, i M_{fi}}_{iT_{fi}}$$

$|S_{fi}|^2$ probability for $i \to f$ transition

Decay rates

- initial state : p^μ, M
- final state : p_1^μ, m_1 ; ... p_i^μ, m_i , ... p_n^μ, m_n

$$i \neq f \Rightarrow |S_{fi}|^2 \propto \left[(2\pi)^4 \delta^{(4)}(P_i - P_f)\right]^2 \to (2\pi)^4 \delta^{(4)}(P_i - P_f) \cdot \underbrace{(2\pi)^4 \delta^{(4)}(0)}_{VT}$$

- $(2\pi)^3 \delta^{(3)}(0) \to V$
- we regularize the time interval by $[-\frac{T}{2}, +\frac{T}{2}]$, then $T \to \infty$

$$|iT_{fi}|^2 = |(2\pi)^4 \delta^{(4)}(P_i - P_f)\, i M_{fi}|^2 = (2\pi)^4 \delta^{(4)}(P_i - P_f)\, VT\, |M_{fi}|^2$$

- we wish to sum over final states :

$$\prod_{i=1}^{n} \sum_{p_i} \longrightarrow \left(\frac{V}{(2\pi)^3}\right)^n \int d^3p_1 \cdots d^3p_i \cdots d^3p_n$$

Probability of a decay in which in the final state the i-th particle has momentum in the interval $[p_i, p_i + dp_i]$:

$$dw = (2\pi)^4 \delta^{(4)}(P_i - P_f)\, VT\, |M_{fi}|^2 \prod_{i=1}^{n} \frac{V d^3p_i}{(2\pi)^3}$$

The above is the probability that the decay take place at any time between $-\frac{T}{2}$ and $\frac{T}{2}$, the probability per unit time

$$d\Gamma = \frac{d\sigma}{T} = (2\pi)^4 \delta^{(4)}(P_i - P_f)\, V \,|M_{fi}|^2 \prod_{i=1}^{n} \frac{V\, d^3 p_i}{(2\pi)^3} \qquad \text{finite for } T \to \infty$$

$$dN = -N \cdot \Gamma\, dt \;\longrightarrow\; \frac{dN}{N} = -\Gamma\, dt \;\rightarrow\; \ln \frac{N(t)}{N_0} = -\Gamma (t - t_0)$$

$$N(t) = N_0\, e^{(t-t_0)/\tau}, \qquad \tau \equiv \Gamma^{-1}$$

Adopting the relativistic normalization we find

$$\underset{\substack{\uparrow \\ (NR)}}{M_{fi}} = \underbrace{\left(\prod_{i=1}^{n} (2E_{p_i} V)^{-1/2}\right)}_{\text{final state}} \underbrace{\left(\prod_{j=1}^{m} (2E_{p_j} V)^{-1/2}\right)}_{\text{initial state}} \underset{\substack{\uparrow \\ (R)}}{\mathcal{M}_{fi}}$$

$$d\Gamma = (2\pi)^4 \delta^{(4)}(P_i - P_f) \cancel{V} \left(\prod_{i=1}^{n} (2E_{p_i} \cancel{V})^{-1} \right) \frac{1}{(2E_p \cancel{V})} |\mathcal{M}_{fi}|^2 \prod_{i=1}^{n} \frac{\cancel{V}\, d^3 p_i}{(2\pi)^3} =$$

$$= (2\pi)^4 \delta^{(4)}(P_i - P_f) \frac{|\mathcal{M}_{fi}|^2}{2E_p} \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3\, 2E_{p_i}}$$

n-body phase-space $d\Phi^{(n)}$:

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$$d\phi^{(n)} = (2\pi)^4 \delta^{(4)}(P_i - P_f) \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2E_i}$$

$$d\Gamma = \frac{1}{2E_p} |\mathcal{M}_{fi}|^2 \, d\phi^{(n)}$$

for $$S_{fi} = \delta_{fi} + (2\pi)^4 \delta^{(4)}(P_i - P_f) \, i \underbrace{\mathcal{M}_{fi}}_{\text{Feynman rules}}$$

- Show that $\frac{d^3 p}{E_p}$ is Lorentz invariant

If there are m identical particles in the final state then configuration that differ by permutation are identical, therefore a factor $\frac{1}{m!}$ must be adopted.

Cross-sections

- beam of particles of mass m_1 with density n_1°, velocity v_1
- target of particles of mass m_2, density n_2° at rest

} uniform distribution

N – number of scattering events, that take place per unit volume and per unit time $\propto$ incoming flux $n^\circ v$ and target density n°

N – number of scattering events, that take place per unit volume and per unit time $\propto$ incoming flux $n_1^\circ v_1$ and target density n_2°

In target rest frame: $$dN = \sigma\,(n_1^\circ v_1)\cdot n_2^\circ \, dV\, dt \qquad [\sigma] = \text{length}^2 = \text{GeV}^{-2}$$

└ cross section definition

In a generic frame (dN should be frame invariant, but a number density changes under the Lorentz transformation as V^{-1})

$$d^4x \longrightarrow dt\, dV$$

$$n_1 n_2 \left[(\bar{v}_1 - \bar{v}_2)^2 - (\bar{v}_1 \times \bar{v}_2)^2 \right]^{1/2} \longrightarrow \underbrace{n_1^\circ v_1 \cdot n_2^\circ}_{\text{target rest frame}}$$

– when colliding particle are collinear then

$$dN = \sigma\, n_1 n_2 \,|\bar{v}_1 - \bar{v}_2|\, dt\, dV$$

$$I \equiv \left[\underbrace{(p_1 \cdot p_2)^2}_{} - m_1^2 m_2^2 \right]^{1/2} = E_1 E_2 \left[(1 - \bar{v}_1\cdot\bar{v}_2)^2 - \frac{m_1^2}{E_1^2}\frac{m_2^2}{E_2^2} \right]^{1/2} = E_1 E_2 \left[(1-\bar{v}_1\cdot\bar{v}_2)^2 - (1-\bar{v}_1^2)(1-\bar{v}_2^2) \right]^{1/2} =$$

└ $(E_1 E_2 - \bar{p}_1\cdot\bar{p}_2)^2 = (E_1 E_2)^2 (1 - \bar{v}_1\cdot\bar{v}_2)^2$

$$\bar{p} = \bar{v}\, E \qquad \frac{m^2}{E^2} = \frac{E^2 - \bar{p}^2}{E^2} = 1 - \frac{\bar{p}^2}{E^2} = 1 - v^2$$

$$= E_1 E_2 \left[\cancel{1} - 2\bar{v}_1\cdot\bar{v}_2 + \underbrace{(\bar{v}_1\cdot\bar{v}_2)^2 - \cancel{1} + \bar{v}_1^2 + \bar{v}_2^2 - \bar{v}_1^2\cdot\bar{v}_2^2}_{-(\bar{v}_1 \times \bar{v}_2)^2} \right]^{1/2} = E_1 E_2 \left[(\bar{v}_1 - \bar{v}_2)^2 - (\bar{v}_1 \times \bar{v}_2)^2 \right]^{1/2}$$

$$= E_1E_2\left[\cancel{1} - 2\bar v_1\cdot\bar v_2 + \underbrace{(\bar v_1\cdot\bar v_2)^2} - \cancel{1} + \bar v_1^2 + \bar v_2^2 - \underbrace{\bar v_1^2\cdot\bar v_2^2}\right] = E_1E_2\left[(\bar v_1-\bar v_2)^2 - (\bar v_1\times\bar v_2)^2\right]$$

$$-(\bar v_1\times\bar v_2)^2$$

$$(\bar v_1\times\bar v_2)^2 = \epsilon_{ijk}v_{1j}v_{2k}\,\epsilon_{ilm}v_{1l}v_{2m} = \underbrace{\epsilon_{jki}\epsilon_{ilm}}_{\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl}}v_{1j}v_{2k}v_{1l}v_{2m} = \bar v_1^2\bar v_2^2 - (\bar v_1\cdot\bar v_2)^2$$

1 2 2 1

$i=3$

1 2 2 1 1 1 2 2 $\to -1$

$\epsilon_{123}\,\epsilon_{321} = -1$

$$dN = \sigma\frac{I}{VE_1E_2}(n_1V)(n_2dV)\,dt$$

$\int n_2\,dV = N_2$, $n_1V = N_1$ — total number of particles

$$\frac{dN}{N_1N_2} = \frac{\sigma I\cancel{T}}{VE_1E_2} = \text{probability of the scattering event} = (2\pi)^4\delta^{(4)}(P_i-P_f)\,V\cancel{T}\int|M_{fi}|^2\prod_{i=1}^{n}\frac{V\,d^3p_i}{(2\pi)^3}$$

$$\sigma = \frac{VE_1E_2}{I}(2\pi)^4\delta^{(4)}(P_i-P_f)\,V\int|M_{fi}|^2\prod_i\frac{V\,d^3p_i}{(2\pi)^3}$$

$$d\sigma = \frac{V^2E_1E_2}{I}(2\pi)^4\delta^{(4)}(P_i-P_f)\,|M_{fi}|^2\prod_i\frac{V\,d^3p_i}{(2\pi)^3}$$

now we switch to (R) normalization

$$|M_{fi}|^2 = \frac{1}{\quad}\,\frac{1}{\quad}\prod^{n}(2E_{p_i}V)^{-1}\,|\mathcal{M}_{fi}|^2$$

$$|M_{fi}|^2 = \underbrace{\frac{1}{2E_1 V}\,\frac{1}{2E_2 V}}_{\text{initial state}} \prod_{i=1}^{n} \left(2E_{p_i} V\right)^{-1} |\mathcal{M}_{fi}|^2$$

$$d\sigma = (2\pi)^4 \delta^{(4)}(P_i - P_f)\, \frac{1}{4I} |\mathcal{M}_{fi}|^2 \underbrace{\prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3\, 2E_{p_i}}}_{d\phi^{(n)}} = \frac{|\mathcal{M}_{fi}|^2}{4I}\, d\phi^{(n)}$$

$$d\sigma = \frac{|\mathcal{M}_{fi}|^2}{4I}\, \frac{d\phi^{(n)}}{m!}$$

for m identical particles in the final state

Two-body final state

- decay:

$$d\phi^{(2)} = \underbrace{(2\pi)^4 \delta^{(4)}(P - p_1 - p_2)}_{\xrightarrow{CM}\ \delta(M - E_1 - E_2)\, \delta^{(3)}(\bar{p}_1 + \bar{p}_2)} \frac{d^3 p_1}{(2\pi)^3\, 2E_1}\, \frac{d^3 p_2}{(2\pi)^3\, 2E_2}$$

$$\int_{dp_2} d\phi^{(2)} = (2\pi)^4 \delta(M - E_1 - E_2)\, \frac{d^3 p_1}{(2\pi)^3\, 2E_1}\, \frac{1}{(2\pi)^3\, 2E_2}\Bigg|_{E_2 \equiv (m_2^2 + \bar{p}_2^{\,2})^{1/2} = (m_2^2 + \bar{p}_1^{\,2})^{1/2}} =$$

$$= \frac{1}{(2\pi)^2}\, \frac{1}{4E_1E_2}\, \delta(M - E_1 - E_2)\, \underbrace{d^3 p_1}_{|\bar{p}_1|^2\, d|\bar{p}_1|\, d\Omega} = \frac{1}{(2\pi)^2}\, d\Omega \int_0^\infty \frac{d|\bar{p}_1|\, |\bar{p}_1|^2}{4E_1E_2}\, \delta\left[M - (m_1^2 + \bar{p}_1^{\,2})^{1/2} - (m_2^2 + \bar{p}_1^{\,2})^{1/2}\right] =$$

$$\delta[f(x)] = \sum_i \frac{\delta(x-x_i)}{|f'(x)|}\Bigg|_{x=x_i,\ f(x_i)=0}$$

$$f(|\bar{p}_1|) = M - (m_1^2 + \bar{p}_1^2)^{1/2} - (m_2^2 + \bar{p}_1^2)^{1/2}$$

$$f(|\bar{p}_1|) = 0 \quad \rightarrow \quad M - (m_1^2 + \bar{p}_1^2)^{1/2} = (m_2^2 + \bar{p}_1^2)^{1/2} \Big|^2$$

$$M^2 - 2M(m_1^2 + \bar{p}_1^2)^{1/2} + m_1^2 + \cancel{\bar{p}_1^2} = m_2^2 + \cancel{\bar{p}_1^2}$$

$$\frac{M^2 + m_1^2 - m_2^2}{2M} = (m_1^2 + \bar{p}_1^2)^{1/2} \Big|^2$$

$$\frac{(M^2 + m_1^2 - m_2^2)^2 - 4M^2 m_1^2}{4M^2} = \bar{p}_1^2 \quad \rightarrow \quad |\bar{p}_1| = + \frac{\left[(M^2 + m_1^2 - m_2^2)^2 - 4M^2 m_1^2\right]^{1/2}}{2M}$$

$$f'(|\bar{p}_1|) = -\frac{1}{\cancel{2}E_1}\cancel{2}|\bar{p}_1| - \frac{1}{\cancel{2}E_2}\cancel{2}|\bar{p}_1| = -|\bar{p}_1|\left(\frac{1}{E_1} + \frac{1}{E_2}\right) = -|\bar{p}_1|\frac{E_1 + E_2}{E_1 E_2} = -|\bar{p}_1|\frac{M}{E_1 E_2}$$

$$d\phi^{(2)} = \frac{d\Omega}{(2\pi)^2}\frac{1}{4\cancel{E_1}\cancel{E_2}}|\bar{p}_1|^{\cancel{2}}\frac{\cancel{E_1}\cdot\cancel{E_2}}{\cancel{|\bar{p}_1|}\,M} = \frac{d\Omega}{32\pi^2 M^2}\underbrace{\left[(M^2 + m_1^2 - m_2^2)^2 - 4M^2 m_1^2\right]}_{M^4 + (m_1^2 - m_2^2)^2 - 2M^2(m_1^2 + m_2^2)}{}^{1/2}$$

if $m_1 = m_2$

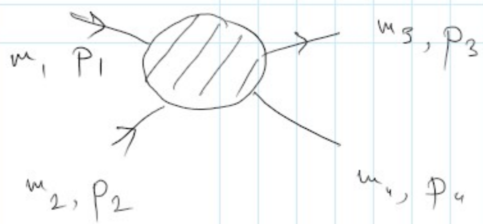
$$d\phi^{(2)} = \frac{1}{32\pi^2}\left(1 - \frac{4m^2}{M^2}\right)^{1/2} d\Omega$$

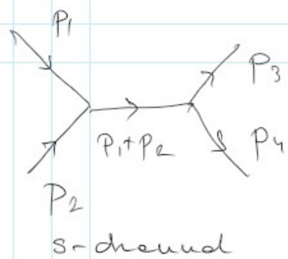
For the decay width one gets

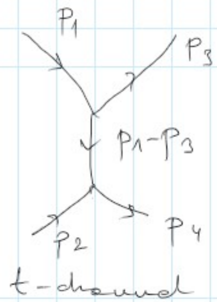
$$d\Gamma = \frac{1}{64\pi^2}\frac{1}{M^3}\left[M^4 + (m_1^2 - m_2^2)^2 - 2M^2(m_1^2+m_2^2)\right]^{1/2} |\mathcal{M}_{fi}|^2 \, d\Omega$$

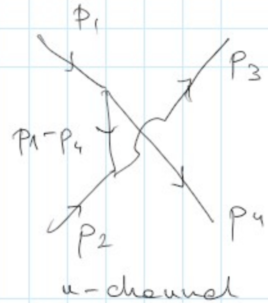
- if there is no external field / force then in $d\Omega = d\cos\theta \, d\varphi$ there is no φ dependence, $d\Omega \to 2\pi \, d\cos\theta$
- if, in addition there is no spin dependence, then $d\cos\theta \to 2$, so $d\Omega \to 4\pi$

- scattering $2 \to 2$









Mandelstam variables:
$$s = (p_1+p_2)^2$$
$$t = (p_1-p_3)^2 \qquad \Rightarrow \qquad s+t+u = \sum_i m_i^2$$
$$u = (p_1-p_4)^2$$

CM: $p_1 = (E_1, \bar{p})$, $p_2 = (E_2, -\bar{p})$, $E_i^2 = m_i^2 + \bar{p}^2$, $s = (E_1+E_2)^2$

$p_3 = (E_3, \bar{p}')$, $p_4 = (E_4, -\bar{p}')$

$$|\bar{p}'| = \frac{1}{2\sqrt{s}}\left[s^2 + (m_3^2 - m_4^2)^2 - 2s(m_3^2+m_4^2)\right]^{1/2} \quad \text{(like for the decay)}$$

$$d\phi^{(2)} = \frac{1}{(2\pi)^2} \frac{|\bar{p}'|}{2\sqrt{s}} \, d\Omega$$

— Find the flux factor I in the CM frame, $I \equiv \left[(p_1 \cdot p_2)^2 - m_1^2 m_2^2\right]^{1/2}$.

$$I = \left[\Big(\underbrace{E_1 E_2 + \bar{p}^2}_{p_1 \cdot p_2}\Big)^2 - m_1^2 m_2^2\right]^{1/2} \qquad E_i = (m_i^2 + \bar{p}^2)^{1/2}$$

$$s = E_1^2 + 2E_1E_2 + E_2^2 = m_1^2 + \bar{p}^2 + 2E_1E_2 + m_2^2 + \bar{p}^2$$

$$E_1 \cdot E_2 = (m_1^2 + \bar{p}^2)^{1/2} (m_2^2 + \bar{p}^2)^{1/2}$$

$$2E_1 \cdot E_2 = s - m_1^2 - m_2^2 - 2\bar{p}^2$$

$$(E_1E_2 + \bar{p}^2)^2 - m_1^2 m_2^2 = (E_1E_2)^2 + 2E_1E_2\bar{p}^2 + \bar{p}^4 - m_1^2 m_2^2 = \cancel{m_1^2 m_2^2} + \bar{p}^2(m_1^2 + m_2^2) + \bar{p}^4 + (s - m_1^2 - m_2^2 - 2\bar{p}^2)\bar{p}^2 + \bar{p}^4 - \cancel{m_1^2 m_2^2} =$$

$$= \bar{p}^2\left(\cancel{m_1^2} + \cancel{m_2^2} + \cancel{\bar{p}^2} + s - \cancel{m_1^2} - \cancel{m_2^2} - \cancel{2\bar{p}^2} + \cancel{\bar{p}^2}\right) = \bar{p}^2 s$$

$$I = |\bar{p}| \, s^{1/2} \qquad d\phi^{(2)} = \frac{1}{(2\pi)^2} \frac{|\bar{p}'|}{2\sqrt{s}} \, d\Omega$$

$$d\sigma = \frac{|\mathcal{M}_{fi}|^2}{4I} d\phi^{(2)} = \frac{|\mathcal{M}_{fi}|^2}{4|\bar{p}| s^{1/2}} \frac{1}{32\pi^2} \frac{|\bar{p}'|}{s^{1/2}} \, d\Omega$$

| different particles in the final state

$$d\sigma = \frac{1}{64\pi^2 s} |\mathcal{M}_{fi}|^2 \frac{|\vec{p}\,'|}{|\vec{p}|} \, d\Omega$$

- in the case of particles with spin we may sum over final spin and average over initial spins

$$|\mathcal{M}_{fi}|^2 \longrightarrow \frac{1}{(2s_a+1)(2s_b+1)} \overbrace{\overline{|\mathcal{M}_{fi}|^2}}^{= \sum_{\text{initial spin}} \sum_{\text{final spin}} |\mathcal{M}_{fi}|^2}$$